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Canonical commutation relation preserving maps

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Abstract

We study maps preserving the Heisenberg commutation relation $ab - ba = 1$. We find a one-parameter deformation of the standard realization of the above algebra in terms of a coordinate and its dual derivative. It involves a non-local 'coordinate' operator while the dual 'derivative' is just the Jackson finite-difference operator. Substitution of this realization into any differential operator involving x and $\frac{\mathrm{d}}{\mathrm{d}x}$ results in an *isospectral* deformation of a continuous differential operator into a finite-difference one. We extend our results to the deformed Heisenberg algebra $ab - qba = 1$. As an example of potential applications, various deformations of the Hahn polynomials are briefly discussed.

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1. Introduction

The Heisenberg algebra[2]

$$[a, b] = 1 \tag{1}$$

made its first appearance in physics, long before the birth of quantum mechanics, through its realization involving a continuous coordinate x and a dual derivative $\mathrm{d}/\mathrm{d}x \equiv \partial$, the latter being the basic differential operator of analysis. General differential operators, in one dimension, are then expressed in terms of powers of ∂, multiplied by functions of x—a wide class of physical problems leads to an eigenvalue equation for such operators.

The reason underlying the predominance of this particular realization in physics is the continuous nature of most spaces under study. Recently, however, there has been a growing interest in discretized versions of spacetime or other, internal, spaces. This sometimes

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[2] We occasionally refer to (1) in what follows as the *canonical commutation relation* (CCR).

originates in the need for numerical computation, as in, e.g., lattice QCD, but has also been proposed as a model of small scale structure. On another front, alternative realizations of (1) have emerged in string theory (see, e.g., [1,9]). In either case, x, ∂ ceases to be the realization of choice and, in several cases, discrete (finite-difference) operators acquire preferred status. Maintaining the validity of (1) makes the transition from the continuous to the discrete non-trivial. The need then arises for new realizations of the Heisenberg algebra in terms of discrete operators. Given such realizations, the differential operators mentioned above can be deformed by replacing the continuous realization by a discrete one—the non-trivial feature of such deformations is that they are isospectral[3]. The process may be regarded as a quantum canonical transformation.

There has been already a considerable amount of research in this direction (see, e.g., [5, 6, 10–12] and references therein), with the discrete derivative ∂_δ, defined by

$$\partial_\delta \triangleright f(x) \equiv \delta^{-1}\big(f(x+\delta) - f(x)\big) \tag{2}$$

traditionally receiving most of the attention. It can be argued that this is due, in part, to the fact that the form of the canonically conjugate 'coordinate' variable x_δ is known (see [10] example 3.3). It is clear, from its definition, that ∂_δ can be restricted to the (equally spaced) points of a lattice. A second natural choice would be an exponential lattice, the corresponding finite-difference operator being the *Jackson derivative* (or *Jackson symbol*, see [4]), defined by

$$\partial_q \triangleright f(x) = \frac{1}{1-q}x^{-1}\big(f(x) - f(qx)\big). \tag{3}$$

The problem with this choice, and the main motivation for this work, is that the form of the canonically conjugate 'coordinate' operator x_q seems to be unknown. We solve this problem in section 2 below while in section 3 we study representations, in a very general setting. Section 4 shows how to generate new canonical commutation relation preserving maps from known ones and section 5 briefly extends the above results to the case of q-canonical commutation relations. In section 6 two isospectral deformations of the Hahn operator are presented as a concrete application—the corresponding polynomial eigenfunctions are also supplied.

2. The Jackson derivative and its canonical conjugate

Given the Jackson derivative ∂_q, satisfying

$$\partial_q x - qx\partial_q = 1 \qquad -1 < q < 1. \tag{4}$$

One finds

$$\partial_q \triangleright x^n = \{n\}x^{n-1} \qquad \{n\} \equiv \frac{1-q^n}{1-q} \quad n = 0, 1, 2, \ldots \tag{5}$$

where

$$\partial_q \triangleright x^n \equiv \partial_q x^n |0\rangle_{x,\partial_q}. \tag{6}$$

The notation used in the rhs of the above equation is as follows. $|0\rangle$ denotes the 'vacuum', a ket annihilated by derivatives, $\partial_q|0\rangle = 0$. The subscript of $|0\rangle$ is an instruction: express all variables to its left in terms of x, ∂_q (already in this form, in this particular example), then use the commutation relation (4) to bring the ∂_q to the right of the x. There they are annihilated by $|0\rangle$ leaving a function of x only—this function serves to define the lhs, i.e. the *action* of ∂_q on x^n. For a general function $f(x)$, defined as a Taylor series in x, the above relation leads

[3] We do not consider here the effect of boundary conditions.

to the alternative definition (3), which makes it clear that ∂_q acts on the exponential lattice $\{x,\ qx,\ q^2x,\ldots\}$. ∂_q can be realized as a pseudodifferential operator

$$\partial_q \sim \frac{1}{1-q}x^{-1}(1-q^A) \equiv x^{-1}\{A\} \qquad A \equiv x\partial \tag{7}$$

where ∂ is the partial derivative w.r.t. x, $\partial x = 1 + x\partial$, also annihilating the vacuum, $\partial|0\rangle = 0$. For the reasons mentioned in the introduction, one would like to realize also an operator x_q, such that $\partial_q x_q = 1 + x_q\partial_q$. Using the commutation relation $Ax^{-1} = x^{-1}(A-1)$ we find $x^{-1}A^n = (A+1)^{n-1}\partial$, so that ($q \equiv \mathrm{e}^h$)

$$\begin{aligned}
\partial_q &= \frac{1}{1-q}x^{-1}(1-q^A)\\
&= -\frac{1}{1-q}x^{-1}\sum_{n=1}^{\infty}\frac{h^n}{n!}A^n\\
&= -\frac{1}{1-q}x^{-1}\sum_{n=1}^{\infty}\frac{h^n}{n!}(1+A)^{n-1}\partial\\
&= -\frac{1}{1-q}(1+A)^{-1}(q^{1+A}-1)\partial\\
\Rightarrow\ \partial_q &= \frac{1}{1-q}B^{-1}(1-q^B)\partial \qquad B \equiv 1+A.
\end{aligned} \tag{8}$$

It will prove convenient in what follows to use the notation $[\![x]\!] \equiv \frac{x}{\{x\}}$, with $[\![0]\!] \equiv 1$. Notice that $\lim_{q\to 1}[\![x]\!] = 1$. We rewrite (8)

$$\partial_q = [\![B]\!]^{-1}\partial \tag{9}$$

∂_q is of the form $\partial_q = f(B)\partial$, $f(B) \equiv [\![B]\!]^{-1}$. We look for x_q in the form $x_q = xg(B)$. Then

$$\begin{aligned}
\partial_q x_q &= f(B)\partial x g(B)\\
&= f(B)g(B) + xf(B+1)g(B+1)\partial.
\end{aligned} \tag{10}$$

The rhs above should be equal to $1 + x_q\partial_q$. One concludes that $g(B) = f(B)^{-1}$, i.e.

$$x_q = x[\![B]\!] = [\![A]\!]x. \tag{11}$$

The action of x_q on monomials is

$$x_q \triangleright x^n = [\![n+1]\!]x^{n+1}. \tag{12}$$

∂_q above acts on power series as a discrete derivative. We examine the corresponding interpretation of the action of x_q. To this end, we introduce the Jackson integral operator S (see, e.g., [4]) given by

$$S \equiv \{A\}^{-1}x. \tag{13}$$

Notice that $\{A\}$ is invertible on the image of x, i.e. on x^n, $n = 1, 2, \ldots$. Comparison with (7) shows that $\partial_q S = 1$ while $(S\partial_q) \triangleright x^n = x^n$, $n = 1, 2, \ldots$ and $(S\partial_q) \triangleright 1 = 0$.

Using the expansion

$$S = (1-q)\sum_{n=0}^{\infty} q^{nA}x \tag{14}$$

one finds

$$S \triangleright f(x) = (1-q)\sum_{n=0}^{\infty} q^n x f(q^n x) \tag{15}$$

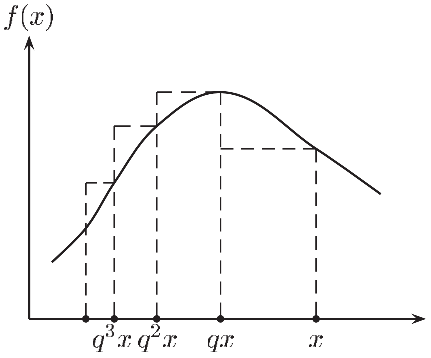

Figure 1. The action of S on f, evaluated at x, gives the area under the dotted lines.

i.e. $S \triangleright f(x)$ gives the area under the dotted lines in figure 1 and converges to $\int_0^x \mathrm{d}x'\, f(x')$ in the limit $q \to 1$. Using the second part of (11), we find

$$x_q = x\partial S. \tag{16}$$

In other words, the action of x_q on $f(x)$ consists in first producing the function $\tilde{f}(x) \equiv (\partial S) \triangleright f(x)$ and then multiplying it by the classical coordinate x.

Aside. We derive an alternative expression for $\tilde{f}(x)$. In classical calculus one has (Rolle theorem)

$$f(x) = \langle f \rangle_x + \langle x f' \rangle_x \tag{17}$$

where $\langle \cdot \rangle_x$ denotes (classical) averaging in the interval $[0, x]$, $\langle f \rangle_x \equiv \frac{1}{x}\int_0^x \mathrm{d}x'\, f(x')$ and $f'(x)$ is the (classical) derivative w.r.t. x. Let now $\langle \cdot \rangle_x^q$ denote *quantum averaging*,

$$\langle f \rangle_x^q \equiv \frac{1}{x} S \triangleright f(x). \tag{18}$$

Then

$$\begin{aligned}(\partial S) \triangleright f(x) &= \partial(1-q)\sum_{n=0}^{\infty} q^n x f(q^n x) \\ &= (1-q)\sum_{n=0}^{\infty} q^n \big(f(q^n x) + x q^n f'(q^n x)\big) \\ &\Rightarrow \quad \tilde{f}(x) = \langle f \rangle_x^q + \langle x f' \rangle_x^q \end{aligned} \tag{19}$$

which is the q-deformed ('quantum') analogue of (17).

Since ∂_q, x_q obey the CCR, one can define a *quantum action* $\triangleright_q$, in complete analogy to the classical one,

$$\partial_q \triangleright_q f(x_q) \equiv \partial_q f(x_q)|0\rangle_{x_q,\partial_q} \tag{20}$$

where, in the rhs, ∂_q is commuted past x_q until it reaches the vacuum, where it gets annihilated—the remaining function of x_q is, by definition, $\partial_q \triangleright f(x_q)$ (notice that the subscript of $|0\rangle$ instructs to express everything in terms of x_q, ∂_q). This extends to arbitrary operators $G(x_q,\, \partial_q)$ acting on functions of x_q, just as in the classical case. It follows trivially that[4]

$$G(x_q,\, \partial_q) \triangleright_q f(x_q) = \phi_q\big(G(x,\, \partial) \triangleright f(x)\big) \tag{21}$$

[4] We have $\phi_q(G(x, \partial)) = G(x_q, \partial_q)$—the ordering of the x and ∂ in $G(x, \partial)$ is immaterial, as long as the same ordering is used in $G(x_q, \partial_q)$.

where $\phi_q : x \mapsto x_q, \partial \mapsto \partial_q$, is the q-deformation map. We note in passing that the quantum averaging operator, $M_q \equiv \frac{1}{x}S$, is the inverse of B. Notice also that the product $x_q\partial_q$ is constant in q (i.e. invariant under the deformation),

$$x_q\partial_q = x\partial = A. \tag{22}$$

This implies that, when dealing with the q-deformation of a general differential operator, terms of the form $x_q^m\partial_q^n$, with $m \leqslant n$, can be expressed entirely in terms of A and ∂_q, the action of which is simpler. The resulting q-deformed operator is a differential-difference operator. This occurs, for example, in the study of the hypergeometric operator. Notice also that the invariance of A permits the exponentiation of the infinitesimal generator of ϕ_q in a trivial manner. Finally, it is worth pointing out that the map ϕ_q admits a non-trivial classical limit, which preserves Poisson brackets. Indeed, with p, q satisfying $\{p, q\} = 1$, where $\{\cdot, \cdot\}$ is the Poisson bracket, one can easily verify that $\{f(A)p, qf(A)^{-1}\} = 1$, where $A \equiv qp$, giving rise to a wide class of classical canonical transformations that might in itself be worth exploring.

3. CCR-preserving maps and adapted bases

Consider the Heisenberg–Weyl universal enveloping algebra $\mathcal{H}$, generated by a, b, with $[a, b] = 1$. In the sequel we work with a certain completion $\hat{\mathcal{H}}$ of $\mathcal{H}$, that allows us to deal with, e.g., exponentials in a, b. From the discussion of the previous section (see (21)), we abstract a map $\phi_q : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ that preserves the CCR[5]

$$\phi_q : a \mapsto a_q \equiv [\![B]\!]^{-1}a \qquad b \mapsto b_q \equiv b[\![B]\!] \tag{23}$$

where B now is $B = 1+ba$. All equations in the previous section depend only on x, ∂ satisfying the CCR and are therefore valid for $(x, \partial) \mapsto (b, a)$. Although we use the particular map ϕ_q given above as an example, we emphasize that our results below are general.

For any pair (a, b) of abstract generators that satisfy the CCR, we say that the set $\{|n\rangle, n = 0, 1, 2, \ldots\}$ is an *adapted basis for* (a, b) if a, b act on it as lowering and raising operators respectively

$$a|n\rangle = n|n-1\rangle \qquad b|n\rangle = |n+1\rangle. \tag{24}$$

Example 3.1 (A classical adapted basis). One particular representation of the Heisenberg algebra is supplied by the subalgebra generated by $\{1, b\}$, an adapted basis being given by $|n\rangle = b^n$. The action of the Heisenberg algebra generators on an arbitrary power series $f(b)$ is $a \triangleright f \equiv [a, f]$ and $b \triangleright f \equiv bf$.

Suppose now we are given a CCR-preserving map ϕ_α, where α denotes any parameters ϕ might depend on, and we wish to find an adapted basis for the deformed generators it produces. A general solution to this problem is possible if we further impose the restriction that ϕ_α be *counit preserving* (i.e. $\phi_\alpha(a)|0\rangle = 0$). It is worth emphasizing that this requirement, although rather natural, nevertheless excludes a number of familiar CCR-preserving maps, like the rotation from x, ∂ to $a^\dagger, a$ in the simple harmonic oscillator. Keeping this observation in mind, we proceed to the following statement: given any CCR *and* counit-preserving map ϕ_α, one can find in general an *induced* map $\tilde{\phi}_\alpha : |n\rangle \mapsto |n\rangle_\alpha$ that maps *any* adapted basis $\{|n\rangle\}$ for (a, b) to an adapted basis $\{|n\rangle_\alpha\}$ for (a_α, b_α). Indeed, to any function $f(b_\alpha)$ one can associate its *b-projection* $\hat{f}(b)$ given by[6]

$$\hat{f}(b) = f(b_\alpha)|0\rangle_{b,a}. \tag{25}$$

[5] We use the same symbol as for the map in (21).
[6] Notice that the b-projection of f depends on the particular deformation ϕ_α used—for simplicity of notation we do not show this dependence explicitly.

We now show that

$$G_\alpha \triangleright \hat{f} = \widehat{G \triangleright f} \tag{26}$$

where $f = f(b)$, $G = G(b, a)$ and $G_\alpha \equiv \phi_\alpha\big(G(b, a)\big) = G(b_\alpha, a_\alpha)$. We have

$$\begin{aligned} G_\alpha \triangleright \hat{f}(b) &\overset{(6)}{=} G_\alpha \hat{f}(b)|0\rangle_{b,a} \\ &\overset{(25)}{=} G_\alpha f(b_\alpha)|0\rangle_{b,a} \\ &\overset{(20)}{=} G_\alpha \triangleright_\alpha f(b_\alpha)|0\rangle_{b,a} \\ &\overset{(21)}{=} \phi_\alpha\,(G \triangleright f(b))\,|0\rangle_{b,a} \\ &= (G \triangleright f)(b_\alpha)|0\rangle_{b,a} \\ &\overset{(25)}{=} \widehat{G \triangleright f}(b). \end{aligned} \tag{27}$$

We comment briefly on the steps that lead to (27). The first equality follows from (6), taking into account that ϕ_α is counit preserving, so that we may put a in place of a_α in the subscript of the vacuum. The second equality follows from (25). In the expression $G_\alpha f(b_\alpha)|0\rangle_{b,a}$ we are instructed to express a_α, b_α in terms of a, b, and then bring the a to the right etc. One can do this in several ways. The one shown above involves first bringing all a_α to the right of the b_α, then substituting $a_\alpha = a_\alpha(b, a)$ and bringing the a to the right (this is equivalent to annihilating the a_α themselves, since ϕ_α preserves the counit). At this point one is left with a function of b_α which is clearly $G_\alpha \triangleright_\alpha f(b_\alpha)$. Finally, one substitutes $b_\alpha = b_\alpha(b, a)$ and brings the a to the right.

Given an adapted basis $\{|n\rangle\}$ for (a, b), we construct the set $\{|n\rangle_\alpha\}$, where[7]

$$|n\rangle_\alpha \equiv \hat{b}^n \tag{28}$$

and claim that it is an adapted basis for (a_α, b_α). Indeed,

$$a_\alpha|n\rangle_\alpha \overset{(28)}{=} a_\alpha \triangleright \hat{b}^n \overset{(27)}{=} \widehat{a \triangleright b^n} = n\widehat{b^{n-1}} = n|n-1\rangle_\alpha. \tag{29}$$

Also,

$$b_\alpha|n\rangle_\alpha = b_\alpha\, b_\alpha^n|0\rangle = b_\alpha^{n+1}|0\rangle = |n+1\rangle_\alpha. \tag{30}$$

Example 3.2 (A quantum adapted basis). Continuing our earlier classical example, we now turn to the realization of the Heisenberg algebra provided by the map ϕ_q. We take $|n\rangle = b^n$ and find for $\tilde{\phi}_q(|n\rangle) \equiv |n\rangle_q$

$$\begin{aligned} |n\rangle_q &= \hat{b}^n \\ &= b_q^n|0\rangle \\ &= (b[\![B]\!])^n|0\rangle \\ &= [\![n]\!]!|n\rangle \end{aligned} \tag{31}$$

where $[\![n]\!]! \equiv [\![1]\!]\,[\![2]\!] \ldots [\![n]\!]$ and $[\![0]\!]! \equiv 1$.

Example 3.3 (A second discrete realization). Consider the pair of operators

$$a_\delta \equiv \phi_\delta(a) = \delta^{-1}(\mathrm{e}^{\delta a} - 1) \qquad b_\delta \equiv \phi_\delta(b) = b\mathrm{e}^{-\delta a}. \tag{32}$$

[7] The vacuum used in computing $\hat{b}^n \equiv b_\alpha^n|0\rangle$ is the ket $|n = 0\rangle$.

One easily verifies that $[a_\delta, b_\delta] = 1$—this is the δ-realization of the CCR mentioned in the introduction. We take again $|n\rangle = b^n$ and compute $\tilde{\phi}_\delta(|n\rangle) \equiv |n\rangle_\delta$

$$\begin{aligned}|n\rangle_\delta &= b_\delta^n|0\rangle \\ &\overset{(32)}{=} b\mathrm{e}^{-\delta a}b\mathrm{e}^{-\delta a}\ldots b\mathrm{e}^{-\delta a}|0\rangle \\ &= b(b-\delta)\cdot\ldots\cdot\big(b-(n-1)\delta\big)\mathrm{e}^{-n\delta a}|0\rangle \\ &= b(b-\delta)\cdot\ldots\cdot\big(b-(n-1)\delta\big) \\ &\equiv b_\delta^{(n)}.\end{aligned} \tag{33}$$

For the *falling δ-factorial polynomials* $b_\delta^{(n)}$, defined by the next-to-last line above (also known as *δ-quasi-monomials* [7]), it holds

$$b_\delta^{(n)} = \sum_{k=1}^{n} s(n,k)\delta^{n-k}b^k \tag{34}$$

where $s(n, k)$ are the Stirling numbers of the first kind.

Example 3.4 (The q-exponential as a b-projection). Consider the spectral problem

$$\partial_q \triangleright f(x) = \lambda f(x).$$

Relying on (26), we look instead at the equation $\partial \triangleright g(x) = \lambda g(x)$ and compute $f(x)$ above from $f = \hat{g}$. We get $g \sim \mathrm{e}^{\lambda x}$ and, using $\widehat{x^n} \overset{(31)}{=} [\![n]\!]x^n$, we find

$$\begin{aligned}f(x) &= \hat{g}(x) \\ &= \mathrm{e}^{\lambda x_q}|0\rangle_{x,\partial} \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}x_q^n|0\rangle_{x,\partial} \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\frac{n!}{\{n\}!}x^n \\ &= \sum_{n=0}^{\infty}\frac{1}{\{n\}!}(\lambda x)^n \\ &\equiv e_q(\lambda x)\end{aligned} \tag{35}$$

i.e. the standard q-deformed exponential [4] is just $e_q(x) = \widehat{\mathrm{e}^x}$. More generally, if

$$a_\alpha = f_\alpha^{-1}(B)a \qquad b_\alpha = bf_\alpha(B)$$

the eigenfunctions of a_α are $\sum_{n=0}^{\infty}\frac{f_\alpha(n)!}{n!}b^n$, where $f_\alpha(n)! \equiv f_\alpha(1)f_\alpha(2)\cdots f_\alpha(n)$, $f_\alpha(0)! \equiv 1$.

It is important to emphasize at this point the formal character of the above results. In particular, a given eigenfunction $g(x)$ of some differential operator may converge for all x, while its b-projection $\hat{g}(x)$ might have a finite (or even zero) radius of convergence as happens, for example, with the δ-exponential

$$e_\delta(x) \equiv \widehat{\mathrm{e}^{\lambda x}} = \sum_{n=0}^{\infty}\frac{1}{n!}x_\delta^{(n)} \tag{36}$$

that appears as the projection $\widehat{\mathrm{e}^{\lambda x}}$ for the map ϕ_δ of example 3.3.

4. Composition of CCR-preserving maps

If ϕ_α, ϕ_β are CCR-preserving maps then so is their composition $\phi_\alpha \circ \phi_\beta$. Considering only smooth maps, with a smooth inverse, one arrives at the notion of the group of CCR-preserving maps (CCR-PM). In the remainder of this paper we impose further the requirement that our maps preserve the counit. We can then use the fact that

$$\widetilde{\phi_\alpha \circ \phi_\beta} = \tilde{\phi}_\alpha \circ \tilde{\phi}_\beta \tag{37}$$

to compute the induced map of a composition of maps. We illustrate this in the following example.

Example 4.1 (Composition of ϕ_q, ϕ_δ). Consider the map ϕ_q discussed earlier. For each value of q, ϕ_q is an element of the CCR-PM. However, $\{\phi_q, \ -1 < q < 1\}$ is not a one-parameter subgroup since, in general, there is no q such that $\phi_{q_1} \circ \phi_{q_2} = \phi_q$. Notice also that $[\phi_{q_1}, \ \phi_{q_2}] \neq 0$, for finite q_1, q_2. Similar remarks hold for ϕ_δ. For the composition $\phi_q \circ \phi_\delta$ we find

$$\begin{aligned} a_{q\delta} &\equiv (\phi_q \circ \phi_\delta)(a) \\ &= \phi_q(\delta^{-1}(\mathrm{e}^{\delta a} - 1)) \\ &= \delta^{-1}\left(\mathrm{e}^{\delta [\![B]\!]^{-1} a} - 1\right) \\ b_{q\delta} &\equiv (\phi_q \circ \phi_\delta)(b) \\ &= \phi_q(b\mathrm{e}^{-\delta a}) \\ &= b[\![B]\!]\mathrm{e}^{-\delta [\![B]\!]^{-1} a} \end{aligned} \tag{38}$$

while for $\phi_\delta \circ \phi_q$ we get

$$\begin{aligned} a_{\delta q} &\equiv (\phi_\delta \circ \phi_q)(a) \\ &= \phi_\delta([\![B]\!]^{-1} a) \\ &= \delta^{-1}[\![1 + \delta^{-1} b(1 - \mathrm{e}^{-\delta a})]\!]^{-1}(\mathrm{e}^{\delta a} - 1) \\ b_{\delta q} &\equiv (\phi_\delta \circ \phi_q)(b) \\ &= \phi_\delta(b[\![B]\!]) \\ &= b\mathrm{e}^{-\delta a}[\![1 + \delta^{-1} b(1 - \mathrm{e}^{-\delta a})]\!]. \end{aligned} \tag{39}$$

For the adapted bases that correspond to the above compositions we find

$$\begin{aligned} |n\rangle_{q\delta} &\equiv \widetilde{\phi_q \circ \phi_\delta}(|n\rangle) \\ &= \phi_q(b_\delta^{(n)}) \\ &= \sum_{k=1}^{n} s(n,k)\delta^{n-k}\phi_q(b^k) \\ &= \sum_{k=1}^{n} s(n,k)\delta^{n-k}[\![k]\!]b^k. \end{aligned} \tag{40}$$

For example,

$$|2\rangle_{q\delta} = \phi_q\,(b(b-\delta)) = [\![2]\!]b^2 - \delta b = \frac{2}{1+q}b^2 - \delta b. \tag{41}$$

Also,

$$\begin{aligned} |n\rangle_{\delta q} &\equiv \widetilde{\phi_\delta \circ \phi_q}(|n\rangle) \\ &= \tilde{\phi}_\delta\,([\![n]\!]!|n\rangle) \\ &= [\![n]\!]!\, b_\delta^{(n)} \end{aligned} \tag{42}$$

so that, for example,

$$|2\rangle_{\delta q} = \frac{2}{1+q} b(b-\delta) \tag{43}$$

which should be compared with (41).

5. Quantum canonical conjugates

Given x, y satisfying the q-Heisenberg algebra

$$xy - qyx = 1 \tag{44}$$

with $-1 < q < 1$. We say that x is the *quantum canonical conjugate* (QCC) of y (and *vice versa*). To complete our treatment of the map ϕ_δ of example 3.3, we undertake here the determination of the QCC of a_δ. We work again with abstract operators a, b and remark that a_q (given in (23)) and b satisfy (44), $a_q b - q b a_q = 1$. Note that the (classical) Heisenberg algebra admits the $*$-involution

$$b^* = a \qquad a^* = b \tag{45}$$

which we extend as complex conjugation to the parameter q, $q^* = q$. Then, taking the $*$ of (44) (with a_q expressed in terms of a, b, as in (9)), we find

$$\begin{aligned} & b^* {a_q}^* - q {a_q}^* b^* = 1 \\ \Rightarrow\ & a\left(b[\![B]\!]^{-1}\right) - q\left(b[\![B]\!]^{-1}\right) a = 1. \end{aligned} \tag{46}$$

Up to now we disposed of the deforming map $\phi_q : a \mapsto a_q, b \mapsto b$, which, applied to a pair of operators satisfying the classical Heisenberg algebra, produces a pair satisfying the quantum Heisenberg algebra. Notice that it does so by leaving b invariant and only deforming a. What we have achieved in (46), is to produce a second similar map ϕ'_q, which instead leaves a invariant and deforms only b: $\phi'_q : a \mapsto a, b \mapsto {a_q}^*$. We only need apply ϕ_δ to (46) to get

$$a_\delta\, \phi_\delta({a_q}^*) - q\phi_\delta({a_q}^*)\, a_\delta = 1 \tag{47}$$

which identifies $\phi_\delta({a_q}^*) = \phi_\delta\left(b[\![B]\!]^{-1}\right)$ as the QCC of a_δ.

6. Deformed Hahn polynomials

As an example of potential applications of our results, we present here various deformations of the Hahn operator and its eigenfunctions, the Hahn polynomials. We start with some definitions. The action of the *Hahn operator* H_δ on a function $f(x)$ is given by

$$\begin{aligned} H_\delta \triangleright f(x) \equiv\ & \delta^{-3}\left(c_4\delta^2 + c_2\delta x + c_1 x^2\right) f(x+\delta) \\ & -\delta^{-3}\left(c_4\delta^2 - \delta(c_1 - 2c_2 + c_3\delta)x + 2c_1 x^2\right) f(x) \\ & -\delta^{-3}\left(\delta(c_1 - c_2 + c_3\delta)x - c_1 x^2\right) f(x-\delta) \end{aligned} \tag{48}$$

where c_i, δ are parameters. What distinguishes H_δ is that it is the most general three-point finite-difference operator with infinitely many polynomial eigenfunctions [10]. The latter are called *Hahn polynomials* (of continuous argument) and we denote them by $h_k^{(\alpha,\beta;\delta)}(x, N)$, where

$$c_2 = N - 2 - \beta \qquad c_3 = -\alpha - \beta - 1 \qquad c_4 = (\beta+1)(N-1) \tag{49}$$

and we have set, without loss of generality, $\delta = 1$ and $c_1 = -1$. The corresponding eigenvalues are

$$\lambda_k = \delta^{-1} c_1 k^2 + c_3 k \qquad k = 0, 1, 2, \ldots. \tag{50}$$

For particular values of their parameters and/or arguments, $h_k^{(\alpha,\beta;\delta)}(x, N)$ reduce to the Meixner, Charlier, Tschebyschev, Krawtchouk or (discrete argument) Hahn polynomials (for details and their rôle in finite-difference equations, see [2, 3, 8] and references therein). We will use the form

$$h_k^{(\alpha,\beta;\delta)}(x, N) = \sum_{i=0}^{k} \gamma_i x_\delta^{(i)} \tag{51}$$

where $x^{(i)}$ is as in (34) and the γ_i are known coefficients.

It has been shown in [10, 11] that H_δ belongs to $\hat{\mathcal{H}}$,

$$H_\delta = c_1 (b_\delta a_\delta)^2 (a_\delta + \delta^{-1}) + c_2 b_\delta a_\delta^2 + c_3 b_\delta a_\delta + c_4 a_\delta \tag{52}$$

where a_δ, b_δ are given in (32), with $(a, b) \mapsto (\partial, x)$. We are now at a setting where our earlier results may be applied directly. First, we deform isospectrally H_δ to H, by effecting the substitution $(a_\delta, b_\delta) \mapsto (\partial, x)$ in (52)—a little bit of algebra gives

$$H = c_1\, x^2 \partial^3 + [(c_1 + c_2) + c_1 \delta^{-1} x]\, x \partial^2 + [c_4 + (c_1 \delta^{-1} + c_3) x]\, \partial. \tag{53}$$

The corresponding polynomial eigenfunctions are obtained directly from (51), using the results of example 3.3,

$$\tilde{h}_k^{(\alpha,\beta)}(x, N) = \sum_{i=0}^{k} \gamma_i x^i \tag{54}$$

(the eigenvalues are, of course, still given by (50)). A second isospectral deformation, involving the substitution $(\partial, x) \mapsto (\partial_q, x_q)$ in (53), leads to

$$H_q = c_1 A^2 (\partial_q + \delta^{-1}) + c_2 A \partial_q + c_3 A + c_4 \partial_q \tag{55}$$

($A = x\partial$) with polynomial eigenfunctions

$$h_k^{(\alpha,\beta;q)}(x, N) = \sum_{i=0}^{k} \gamma_i [\![i]\!]!\, x^i. \tag{56}$$

Note the ease with which $h_k^{(\alpha,\beta;q)}(x, N)$ are computed, despite the highly non-trivial complexity of the differential-difference operator H_q. Finally, we point out that the *spectrum* of H in (53) may also be q-deformed by effecting the substitution $(\partial, x) \mapsto (\partial_q, x)$ in (53)—one gets a finite-difference operator

$$\tilde{H}_q = c_1\, x^2 {\partial_q}^3 + (c_1 + c_2 + c_1 \delta^{-1} x)\, x {\partial_q}^2 + [c_4 + (c_1 \delta^{-1} + c_3) x]\, \partial_q \tag{57}$$

with infinitely many polynomial eigenfunctions—the corresponding eigenvalues are

$$\tilde{\lambda}_k = c_1 \delta^{-1} \{k\} \big(\{k-1\} + 1 \big) + c_3 \{k\} \qquad k = 0, 1, 2, \ldots. \tag{58}$$

Note added in proof. After this work was sent for publication, Professor C Zachos, to whom we express our gratitude, pointed out to us that x_q, ∂_q can be related to x, ∂ via a similarity transformation. Indeed, starting with the ansatz $U(A)^{-1} \partial U(A) = \partial_q$, it follows that $\partial U(A) = \partial_q U(A-1)$ and, using (7), $U(A) = [\![A]\!]^{-1} U(A-1)$, from which the formal expression $U(A) = \Gamma_q(A+1)/\Gamma(A+1)$ can be derived. Here Γ_q denotes the q-deformed gamma function, $\Gamma_q(x+1) = \{x\}\Gamma_q(x)$ (see, e.g., [4]). Then, x_q is computed as $x_q = U(A)^{-1} x U(A) = U(A)^{-1} U(A-1) x = [\![A]\!] x$ and the constancy of A in q follows trivially.

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